CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2019 Trial Examination

FORM VI

MATHEMATICS EXTENSION 1

Monday 19th August 2019

General Instructions

- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black pen.
- NESA-approved calculators and templates may be used.

Total — 70 Marks

- All questions may be attempted.

Section I — 10 Marks

- Questions 1 – 10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II — 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference Sheet
- Candidature — 143 boys

Examiner

SDP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Which of the following is an even function?

(A) $y = x$

(B) $y = 2^x$

(C) $y = (x-2)^4$

(D) $y = \sqrt{5 - x^2}$

QUESTION TWO

Which of the following is equal to $\displaystyle\int \frac{dx}{4 + x^2}$?

(A) $\cos^{-1} 2x + C$

(B) $2\sin^{-1} x + C$

(C) $\dfrac{1}{2}\tan^{-1}\dfrac{x}{2} + C$

(D) $\log_e(1 + x^2) + C$

QUESTION THREE

Which of the following is equal to $\dfrac{100!}{98! \times 2!}$?

(A) $100 \times 99 \times 98$

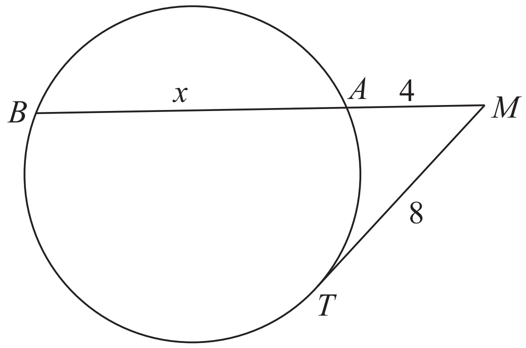
(B) 100×99

(C) 50×99

(D) $50 \times 49\frac{1}{2}$

Examination continues next page ...

QUESTION FOUR



Suppose TM is a tangent to a circle at T, while MB is a secant intersecting the circle at A and B. Given that $TM = 8$, $AB = x$ and $MA = 4$, what is the value of x?

(A) $2\sqrt{17} - 2$

(B) 12

(C) 14

(D) 16

QUESTION FIVE

Which of the following is the primitive of $\cos^2 x$?

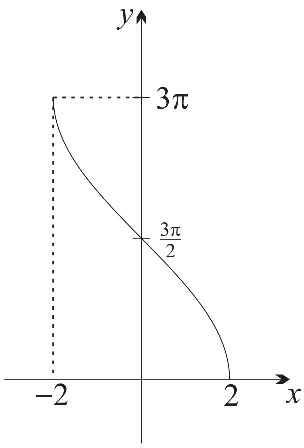
(A) $x + \frac{1}{2}\cos 2x + C$

(B) $x - \frac{1}{2}\cos 2x + C$

(C) $\frac{1}{2}x + \frac{1}{4}\sin 2x + C$

(D) $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$

QUESTION SIX



Which equation is best represented by the graph above?

(A) $y = 3\cos^{-1}\left(\frac{x}{2}\right)$

(B) $y = 6\sin^{-1}\left(\frac{x}{2}\right)$

(C) $y = \frac{3}{2}\cos^{-1}(2x)$

(D) $y = 2\sin^{-1}(x)$

QUESTION SEVEN

Which of the following polynomials are divisible by $x + 1$?

(I) $x^{2019} - 1$ (II) $x^{2019} + 1$ (III) $x^{2020} - 1$ (IV) $x^{2020} + 1$

(A) (I) and (III) only

(B) (II) and (IV) only

(C) (II) and (III) only

(D) (I) and (IV) only

QUESTION EIGHT

Which of the following equations is true, given that $\ddot{x} = 2x(3x - 1)$?

(A) $v = 2x^3 - x^2 + C$

(B) $v^2 = 2x^3 - x^2 + C$

(C) $v = x^2(x^3 - x) + C$

(D) $v^2 = 4x^3 - 2x^2 + C$

Examination continues next page ...

QUESTION NINE

What is the derivative of $y = \sqrt{1+\sqrt{x}}$?

(A) $\dfrac{1}{2\sqrt{1+\sqrt{x}}}$

(B) $\dfrac{1}{\sqrt{x}\sqrt{1+\sqrt{x}}}$

(C) $\dfrac{1}{2\sqrt{x}\sqrt{1+\sqrt{x}}}$

(D) $\dfrac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}$

QUESTION TEN

What is the value of $\tan(\alpha+\beta)$ if $\tan\alpha + \tan\beta + 4 = \cot\alpha + \cot\beta = 10$?

(A) $\dfrac{3}{5}$

(B) $\dfrac{5}{3}$

(C) 6

(D) 15

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\lim_{x \to 0} \dfrac{\sin 2x}{3x}$. **1**

(b) Find the domain of the function $y = 4\sin^{-1}(2x - 3)$. **1**

(c) The equation $x^3 + 6x^2 - 2x + 4 = 0$ has roots α, β and γ. Find the value of:

(i) $\alpha + \beta + \gamma$ and $\alpha\beta + \beta\gamma + \gamma\alpha$, **1**

(ii) $\alpha^2 + \beta^2 + \gamma^2$. **2**

(d) The volume of water in a lake increases over time. The flow of water into the lake is given by the flow rate $\dfrac{dV}{dt} = 120\,(3 - \sin 2t)$ where V is the volume of water in the lake in cubic metres at time t in days.

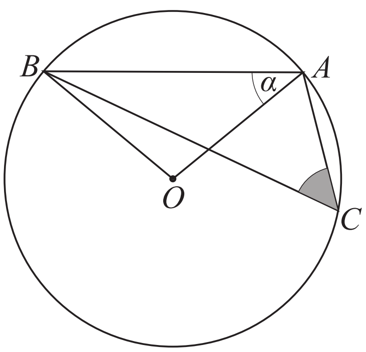
(i) What is the maximum flow rate of water? **1**

(ii) Given that the lake has initial volume of $5000\,\text{m}^3$ find V in terms of t. **2**

(e) Differentiate $y = \tan^{-1}(\log_e x)$. Give your answer in simplest form. **2**

(f) Evaluate $\displaystyle\int_e^{e^3} \frac{dx}{2x\ln x}$ using the substitution $u = \ln x$. Give your answer in exact form. **3**

(g)



In the diagram above O is the centre of the circle. Points A, B and C all lie on the circumference of the circle. If $\angle OAB = \alpha$ find the size of $\angle ACB$. Give reasons for your answer. **2**

Examination continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Solve $\dfrac{3}{x} < 2$. **2**

(b) Prove that $\ddot{x} = \dfrac{d}{dx}\left(\frac{1}{2}v^2\right)$. **1**

(c) A particle P is moving in a straight line with its motion given by $\ddot{x} = -16x$ where x is the displacement of P from the origin O. Initially P is 3 metres on the right of O and is moving towards O with velocity $4\sqrt{3}\,\text{m/s}$.

(i) Show that the speed of the particle is given by $4\sqrt{12 - x^2}\,\text{m/s}$. **2**

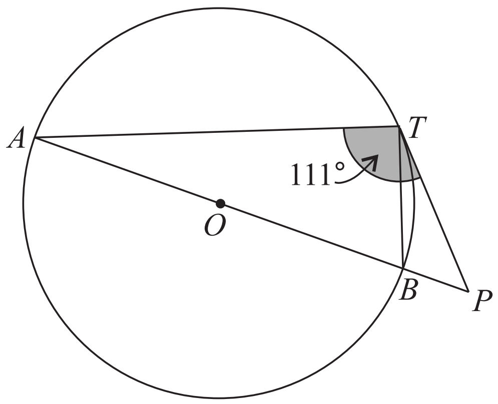
(ii) Verify that $x = A\sin(4t + \alpha)$ satisfies $\ddot{x} = -16x$ for all values of the constants A and α. **1**

(d) Let $f(x) = \frac{1}{5}x - \log_e x$.

(i) Show that $f(x)$ has a root between $x = 1$ and $x = e$. **1**

(ii) Taking $x_1 = 1{\cdot}5$ as an initial approximation, use Newton's method once to obtain x_2, a better approximation of the root. Write down the value of x_2 correct to two decimal places. **2**

(e)



In the diagram above AB is a diameter of the circle, TP is a tangent at point T, O is the centre of the circle and $\angle ATP = 111°$. Find $\angle BAT$ giving reasons. **2**

(f) A chord PQ joins the points $P\left(2p, p^2\right)$ and $Q\left(2q, q^2\right)$ on the parabola $x^2 = 4y$. The chord PQ passes through the point $A(0, 2)$.

(i) Derive the equation of the chord PQ. **1**

(ii) Find the coordinates of M, the midpoint of PQ. **1**

(iii) Show that $pq = -2$. **1**

(iv) Hence find the equation of the locus of M as P and Q vary. **1**

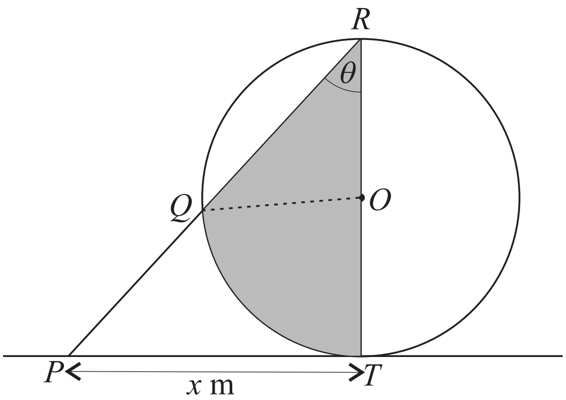
Examination continues overleaf ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Using $t = \tan\frac{\theta}{2}$ show that $\sin\theta + \cos\theta = \frac{1}{4}$ can be written as $5t^2 - 8t - 3 = 0$. **2**

(ii) Hence solve $\sin\theta + \cos\theta = \frac{1}{4}$ for $-\pi < \theta < \pi$. Give your answer correct to one decimal place. **2**

(b)



The above diagram shows a circle centre O, with radius 1 metre. The line PT of length x m is a tangent to the circle at T and RT is a diameter. The line PR cuts the circle at Q. Let $A\,\text{m}^2$ be the area of the shaded region and let $\angle ORQ = \theta$ in radians. The point P is moving away from T at a constant speed of 16 m/s.

(i) Express $\tan\theta$ in terms of x and find θ when $x = \dfrac{2}{\sqrt{3}}$. **1**

(ii) Find $\dfrac{d\theta}{dt}$ when $x = \dfrac{2}{\sqrt{3}}$. **2**

(iii) Show that $A = \theta + \dfrac{1}{2}\sin 2\theta$. **2**

(iv) Find $\dfrac{dA}{dt}$ when $x = \dfrac{2}{\sqrt{3}}$. **2**

(c) (i) Let t_r be the coefficient of x^r in the expansion of $(a + bx)^n$. Show that: **2**

$$\frac{t_{r+1}}{t_r} = \frac{n-r}{r+1} \times \frac{b}{a}.$$

(ii) Hence, or otherwise, find the coefficients of the two consecutive terms that have equal coefficients in the expansion of $(2 + 3x)^{14}$. **2**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) The polynomial $P(x)$ is divided by $(x+2)(x-5)$. Find the remainder given that $P(-2) = 6$ and $P(5) = -1$. **2**

(b) A particle is projected from the origin on level ground with speed 15 m/s at an angle α to the horizontal. Let the acceleration due to gravity be $g = 10\,\text{m/s}^2$.

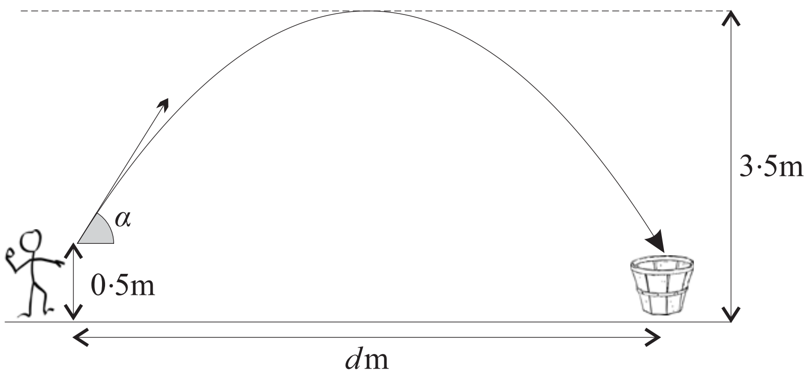
(i) Derive the equations for x and y, the horizontal and vertical displacement of the particle respectively in terms of t. **1**

(ii) Show that the maximum height reached by the particle h metres is given by **2**

$$h = \frac{45}{4}\sin^2\alpha\,.$$

(iii) Show that the particle returns to the initial height at $x = \dfrac{45}{2}\sin 2\alpha\,.$ **2**

(iv) Sophie throws a paper ball into the centre of a bin across a room. The paper ball is projected from a point 0·5 m above the floor and the top of the bin is also 0·5 m above the floor. The ceiling height is 3·5 m above the floor.



The paper ball is thrown with a velocity 15 m/s at an angle of α. Assuming no air resistance, show that the maximum separation d metres that Sophie and the bin can have and still get the paper ball into the bin is $d = 6\sqrt{11}$ m. **2**

(c) Use mathematical induction to show that for any integer $n \geq 0$, **3**

$$\sum_{r=0}^{n} \frac{1}{2^r}\tan\left(\frac{x}{2^r}\right) = \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right) - 2\cot(2x)\,,$$

where $0 < x < \frac{\pi}{4}$.

(d) If the roots of the quadratic equation $8x^2 - 5x + a = 0$ are $\sin\theta$ and $\cos 2\theta$ for some angle θ, find the possible values of a. **3**

End of Section II

END OF EXAMINATION

Extension 1 Trial Solutions 2019

1. (D) ✓

   (A) $f(-x) = -x$ not even
   (B) $f(-x) = 2^{-x}$ not even
   (C) $f(-x) = (-x-2)^4 = (x-2)^4$ not even
   (D) $f(-x) = \sqrt{5-(-x)^2} = \sqrt{5-x^2}$ so even

2. (C) ✓

$$\int \frac{1}{4+x^2}dx = \int \frac{1}{2^2+x^2}dx$$
$$= \tfrac{1}{2}\tan^{-1}\tfrac{x}{2} + C$$

3. (C) ✓

$$\frac{100!}{98! \times 2!} = \frac{100 \times 99 \times 98!}{98! \times 2}$$
$$= \frac{100 \times 99}{2}$$
$$= 50 \times 99$$

4. (B) ✓

$$AM \times MB = TM^2 \qquad \text{(tangent and secant)}$$
$$4 \times (x+4) = 8^2$$
$$4x + 16 = 64$$
$$4x = 48$$
$$x = 12$$

5. (C) ✓

$$\int \cos^2 x\, dx = \frac{1}{2}\int (\cos 2x + 1)\, dx \qquad \text{(from double angle formula)}$$
$$= \frac{1}{2}\left(\frac{1}{2}\sin 2x + x\right) + C$$
$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

6. (A) ✓
   $\cos^{-1} x$ domain is $-1 \le x \le 1$.
   $\cos^{-1} x$ range is $0 \le y \le \pi$.
   so this graph must be $y = 3\cos^{-1}\left(\frac{x}{2}\right)$.

7. (C) ✓

If divisible by $(x+1)$ then $P(-1)=0$

$$\begin{aligned}\text{(I) } P(-1) &= ((-1)^{2019}-1)\\ &= -1-1\\ &= -2 \qquad \text{so not divisible by } x+1\end{aligned}$$

$$\begin{aligned}\text{(II) } P(-1) &= ((-1)^{2019}+1)\\ &= -1+1\\ &= 0 \qquad \text{so divisible by } x+1\end{aligned}$$

$$\begin{aligned}\text{(III) } P(-1) &= ((-1)^{2020}-1)\\ &= 1-1\\ &= 0 \qquad \text{so divisible by } x+1\end{aligned}$$

$$\begin{aligned}\text{(IV) } P(-1) &= ((-1)^{2020}+1)\\ &= 1+1\\ &= 2 \qquad \text{so not divisible by } x+1\end{aligned}$$

8. (D) ✓

$$\begin{aligned}\tfrac{d}{dx}(\tfrac{1}{2}v^2) &= 6x^2-2x\\ \tfrac{1}{2}v^2 &= 2x^3-x^2+\tfrac{1}{2}C\\ v^2 &= 4x^3-2x^2+C\end{aligned}$$

9. (D) ✓

$$\begin{aligned}y &= \left(1+x^{\frac{1}{2}}\right)^{\frac{1}{2}}\\ \tfrac{dy}{dx} &= \tfrac{1}{2}\times\tfrac{1}{2}x^{-\frac{1}{2}}\times\left(1+x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\\ &= \frac{1}{2\times 2\times\sqrt{x}\times\sqrt{1+\sqrt{x}}}\\ &= \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}\end{aligned}$$

10. (D) ✓

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\begin{aligned} \tan\alpha + \tan\beta + 4 &= 10 \\ \tan\alpha + \tan\beta &= 6 \end{aligned}$$

$$\begin{aligned} \cot\alpha + \cot\beta &= 10 \\ \frac{1}{\tan\alpha} + \frac{1}{\tan\beta} &= 10 \\ \frac{\tan\alpha + \tan\beta}{\tan\alpha\tan\beta} &= 10 \\ \frac{6}{\tan\alpha\tan\beta} &= 10 \\ \tan\alpha\tan\beta &= \frac{3}{5} \end{aligned}$$

$$\begin{aligned} \tan(\alpha+\beta) &= \frac{6}{1 - \frac{3}{5}} \\ &= 15 \end{aligned}$$

11. (a)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{3x} &= \frac{2}{3}\times\lim_{x\to 0}\frac{\sin 2x}{2x}\\ &= \frac{2}{3}\times 1\\ &= \frac{2}{3} \quad \checkmark\end{aligned}$$

(b) Domain of $4\sin^{-1}x$ is $-1 \le x \le 1$
Domain of $4\sin^{-1}(2x-3)$ is:

$$\begin{aligned}-1 \le 2x-3 \le 1\\ 2 \le 2x \le 4\\ 1 \le x \le 2 \quad \checkmark\end{aligned}$$

(c) i. $\alpha+\beta+\gamma = -6$ and $\alpha\beta+\beta\gamma+\gamma\alpha = -2$ ✓ (both)

ii.

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2 &= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) \quad \checkmark\\ &= (-6)^2 - 2\times -2\\ &= 40 \quad \checkmark\end{aligned}$$

(d) i. Maximum flow rate occurs when $\sin 2t = -1$
So when $\sin 2t = -1$,

$$\frac{dV}{dt} = 120(3-(-1))$$
$$\frac{dV}{dt} = 480\,\mathrm{m}^3/day \quad \checkmark$$

ii.

$$\begin{aligned}\frac{dV}{dt} &= 120\,(3-\sin 2t)\\ V &= 120\left(3t+\tfrac{1}{2}\cos 2t\right)+C \quad \checkmark\end{aligned}$$

$V = 5000$ when $t = 0$

$$\begin{aligned}5000 &= 120\times\tfrac{1}{2}+C\\ C &= 4940\\ V &= 120\left(3t+\tfrac{1}{2}\cos 2t\right)+4940 \quad \checkmark \text{ (or similar)}\end{aligned}$$

(e)

$$\begin{aligned}f(x) &= \tan^{-1}(\log_e x)\\ f'(x) &= \frac{1}{x}\times\frac{1}{1+(\log_e x)^2} \quad \checkmark\\ &= \frac{1}{x\,(1+(\log_e x)^2)} \quad \checkmark \text{ (or similar)}\end{aligned}$$

(f)

$$\int_e^{e^3} \frac{1}{2x\ln x}dx = \int_1^3 \frac{1}{2u}du \quad \checkmark \qquad u = \ln x \qquad x = e^3, u = 3$$

$$= \frac{1}{2}\left[\ln u\right]_1^3 \quad \checkmark \qquad \frac{du}{dx} = \frac{1}{x} \qquad x = e, u = 1$$

$$= \frac{1}{2}\left[\ln 3 - \ln 1\right] \qquad du = \frac{1}{x}dx$$

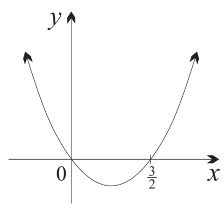
$$= \frac{1}{2}\ln 3 \quad \checkmark$$

(g) $OB = OA$ (radii) so $\triangle OAB$ is isosceles
$\angle OBA = \alpha$ (equal base angles in isosceles triangle)
$\angle AOB = 180^\circ - 2\alpha$ (angles in a triangle) ✓

$\angle ACB = \frac{1}{2} \times (180^\circ - 2\alpha)$ (angle at circumference is halve angle at centre)
$= 90^\circ - \alpha$ ✓ (must have reasons for both marks)

12. (a)

$\frac{3}{x} < 2$ multiply both sides by x^2
$3x < 2x^2$ ✓
$0 < x(2x - 3)$
from graph, $x < 0$ or $x > \frac{3}{2}$ ✓



(b)

$$RHS = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$$
$$= \frac{d}{dv}\left(\tfrac{1}{2}v^2\right) \times \frac{dv}{dx}$$
$$= v\frac{dv}{dx}$$
$$= \frac{dx}{dt} \times \frac{dv}{dx}$$
$$= \frac{dv}{dt}$$
$$= \ddot{x} \quad \checkmark \text{ (show that question)}$$
$$= LHS$$

(c) i.

$$\begin{aligned}
\ddot{x} &= -16x \\
\frac{d}{dx}\left(\tfrac{1}{2}v^2\right) &= -16x \\
\tfrac{1}{2}v^2 &= \tfrac{-16}{2}x^2 + \tfrac{1}{2}C \\
v^2 &= -16x^2 + C
\end{aligned}$$

initially, $x = 3$ and $v = 4\sqrt{3}$

$$\begin{aligned}
\left(4\sqrt{3}\right)^2 &= -16 \times 3^2 + C \\
48 &= -144 + C \\
C &= 192 \quad ✓
\end{aligned}$$

$$\begin{aligned}
\text{so } v^2 &= 192 - 16x^2 \\
|v| &= \sqrt{192 - 16x^2} \qquad \text{(as speed is positive)} \\
&= 4\sqrt{12 - x^2} \qquad ✓ \text{ (show that question)}
\end{aligned}$$

ii.

$$\begin{aligned}
x &= A\sin(4t + \alpha) \\
v &= 4A\cos(4t + \alpha) \\
\ddot{x} &= -16A\sin(4t + \alpha) \\
&= -16x \quad ✓
\end{aligned}$$

(d) i. $f(1) = \frac{1}{5} - \log_e 1 = \frac{1}{5}$
$f(e) = \frac{1}{5} \times e - \log_e e \approx -0.46$
Therefore as $f(1)$ is positive and $f(e)$ is negative, it has a root between 1 and e as $f(x)$ is continuous. ✓(both values found)

ii. $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$f'(x) = \frac{1}{5} - \frac{1}{x}$ ✓

$$\begin{aligned}
x_2 &= 1.5 - \frac{\frac{1}{5} \times 1.5 - \log_e 1.5}{\frac{1}{5} - \frac{1}{1.5}} \\
&\approx 1.27 \quad ✓
\end{aligned}$$

(e) $\angle ATB = 90^\circ$ (Thales Theorem)
$\angle BTP = 111^\circ - 90^\circ = 21^\circ$ (adjacent angles) ✓
$\angle BAT = 21^\circ$ (alternate segment theorem) ✓

(f) i. Gradient of line $PQ = \frac{p^2-q^2}{2p-2q} = \frac{(p-q)(p+q)}{2(p-q)} = \frac{p+q}{2}$

Equation of chord PQ:

$$\begin{aligned}
y - p^2 &= \frac{p+q}{2}(x - 2p) \\
y &= \frac{p+q}{2}x - pq \qquad ✓ \text{ (or similar)}
\end{aligned}$$

ii.

$$M = \left(\frac{2p+2q}{2}, \frac{p^2+q^2}{2}\right)$$
$$= \left(p+q, \frac{p^2+q^2}{2}\right) \quad \checkmark$$

iii.

chord passes through $A(0,2)$

$$2 = -pq$$
$$-2 = pq \quad \checkmark \text{ (show that question)}$$

iv. From, 12bii, $x = p+q$ and $y = \frac{p^2+q^2}{2}$

$$y = \frac{(p+q)^2 - 2pq}{2}$$
$$= \frac{x^2 - 2(-2)}{2}$$
$$y = \tfrac{1}{2}x^2 + 2 \quad \checkmark$$

13. (a) i. When $t = \tan\frac{\theta}{2}$, $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$

$$\sin\theta + \cos\theta = \tfrac{1}{4}$$
$$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = \tfrac{1}{4} \quad \checkmark$$
$$\frac{2t+1-t^2}{1+t^2} = \tfrac{1}{4}$$
$$8t + 4 - 4t^2 = 1 + t^2$$
$$5t^2 - 8t - 3 = 0 \quad \checkmark \text{ (show that question)}$$

ii.

$$5t^2 - 8t - 3 = 0$$
$$t = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 5 \times -3}}{2 \times 5}$$
$$t = \frac{4 \pm \sqrt{31}}{5} \quad \checkmark$$
$$\text{so } \tan\frac{\theta}{2} = \frac{4 \pm \sqrt{31}}{5}$$
$$\frac{\theta}{2} \approx 1.08924 \text{ or } \frac{\theta}{2} \approx -0.30384$$
$$\theta \approx 2.2 \text{ or } \theta \approx -0.6 \quad \checkmark \text{ (both)}$$

(b) i. $\tan\theta = \frac{x}{2}$

When $x = \frac{2}{\sqrt{3}}$, $\theta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ ✓

ii. So $\frac{dx}{dt} = 16$

$\frac{dx}{d\theta} = 2\sec^2\theta$ and $\frac{d\theta}{dx} = \frac{1}{2}\cos^2\theta$ ✓

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{d\theta}{dx} \times \frac{dx}{dt}\\ &= \frac{1}{2}\cos^2\theta \times 16\\ &= \frac{1}{2}\cos^2\frac{\pi}{6} \times 16\\ &= 6 \text{ rad/s} \quad ✓\end{aligned}$$

iii. In $\triangle ORQ$, $OR = OQ = 1$m (radii) and $\triangle ORQ$ is isosceles so $OQR = \theta$ (base angles of isosceles triangles are equal) and $\angle ROQ = 180^\circ - 2\theta$ (angles in a traingle).

Area $\triangle ORQ = \frac{1}{2} \times 1 \times 1 \times \sin(180^\circ - 2\theta) = \frac{1}{2}\sin 2\theta$ ✓

$\angle QOT = 2\theta$ (angle at centre double angle at circumference)
Area Sector $OQT = \frac{1}{2} \times 1^2 \times 2\theta = \theta$ ✓

Therefore $A = \theta + \frac{1}{2}\sin 2\theta$

iv.

$$\begin{aligned}\frac{dA}{d\theta} &= 1 + \cos 2\theta \quad ✓\\ \frac{dA}{dt} &= \frac{dA}{d\theta} \times \frac{d\theta}{dt}\\ &= (1 + \cos 2\theta) \times 6\\ &= \left(1 + \cos\frac{\pi}{3}\right) \times 6\\ &= 9\,\text{m}^2/\text{s} \quad ✓\end{aligned}$$

(c) i.

$$(a+bx)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r}(bx)^r$$

$$\begin{aligned} t_r &= {}^nC_r\, a^{n-r}b^r \\ t_r &= {}^nC_{r+1}\, a^{n-r-1}b^{r+1} \end{aligned}$$

$$\frac{t_{r+1}}{t_r} = \frac{{}^nC_{r+1}\, a^{n-r-1}b^{r+1}}{{}^nC_r\, a^{n-r}b^r} \qquad \checkmark$$

$$= \frac{\frac{n!}{(n-r-1)!(r+1)!}a^{n-r-1}b^{r+1}}{\frac{n!}{(n-r)!r!}a^{n-r}b^r}$$

$$= \frac{n!(n-r)!r!}{n!(n-r-1)!(r+1)!} \times \frac{b}{a} \qquad \checkmark$$

$$= \frac{n-r}{r+1} \times \frac{b}{a}$$

ii. Equal coefficients when $\frac{t_{r+1}}{t_r} = 1$. $a = 2$, $b = 3$ and $n = 14$, so

$$\begin{aligned} 1 &= \frac{n-r}{r+1} \times \frac{b}{a} \\ b(n-r) &= a(r+1) \\ 3 \times (14-r) &= 2(r+1) \\ 42 - 3r &= 2r + 2 \\ 40 &= 5r \\ r &= 8 \qquad \checkmark \end{aligned}$$

when $r = 8$

$$\begin{aligned} t_r &= {}^{14}C_8\, 2^{14-8}\, 3^8 \\ &= 1\,260\,971\,712 \qquad \checkmark \end{aligned}$$

14. (a) When $P(x)$ is divided by $(x+2)(x-5)$ it may have a linear remainder so we can write:

$P(x) = (x+2)(x-5)Q(x) + Ax + B$,
where $Q(x)$ is a polynomial and A and B are constants.

$P(-2) = 6 = -2A + B$ (1)
$P(5) = -1 = 5A + B$ (2) ✓ (both)

(2) − (1): $-7 = 7A$ so $A = -1$ and $B = 4$

So the remainder is $4 - x$ ✓

(b) i.

$$\begin{aligned} \ddot{x} &= 0 & \ddot{y} &= -10 \\ \dot{x} &= C_1 & \dot{y} &= -10t + C_3 \\ \text{when } t = 0,\ \dot{x} &= 15\cos\alpha & \text{when } t = 0,\ \dot{y} &= 15\sin\alpha \\ \dot{x} &= 15\cos\alpha & \dot{y} &= -10t + 15\sin\alpha \\ x &= 15t\cos\alpha + C_2 & y &= -5t^2 + 15t\sin\alpha + C_4 \\ \text{when } t = 0,\ x &= 0\cos\alpha & \text{when } t = 0,\ y &= 0 \\ x &= 15t\cos\alpha & y &= -5t^2 + 15t\sin\alpha \quad \checkmark \text{ (both)} \end{aligned}$$

ii. Maximum height when $\dot{y} = 0$, so:

$$\begin{aligned} 0 &= -10t + 15\sin\alpha \\ t &= \tfrac{3}{2}\sin\alpha \quad \checkmark \end{aligned}$$

Maximum height at $t = \frac{3}{2}\sin\alpha$ and $y = h$:

$$\begin{aligned} h &= -5\left(\tfrac{3}{2}\sin\alpha\right)^2 + 15 \times \tfrac{3}{2}\sin\alpha \times \sin\alpha \\ h &= -\tfrac{45}{4}\sin^2\alpha + \tfrac{45}{2}\sin^2\alpha \\ h &= \tfrac{45}{4}\sin^2\alpha \quad \checkmark \text{ (show that question)} \end{aligned}$$

iii. As the motion is symmetrical, it returns back to the inital height at twice the time taken to reach maximum height. Therefore it returns to the initial height at $t = 3\sin\alpha$. ✓

When $t = 3\sin\alpha$:

$$\begin{aligned} x &= 15 \times 3\sin\alpha \times \cos\alpha \\ &= 45\sin\alpha\cos\alpha \\ &= \tfrac{45}{2}\sin 2\alpha \quad \checkmark \text{ (show that question)} \end{aligned}$$

iv. Taking the point of projection as the origin, the paper bin is at the same height. Therefore the maximum height that is possible is 3m.

From 14bi)

$$\begin{aligned} 3 &= \tfrac{45}{4}\sin^2\alpha \\ \tfrac{4}{15} &= \sin^2\alpha \\ \sin\alpha &= \pm\tfrac{2\sqrt{15}}{15} \\ \text{as the angle of projection is positive, } \sin\alpha &= \tfrac{2\sqrt{15}}{15} \quad \checkmark \text{ (or similar)} \end{aligned}$$

the maximum distance is given when $\alpha = \frac{\pi}{4}$ as this α is less than $\frac{\pi}{4}$ it must be the maximum distance possible.

subsitute this into the formula in 14bii):

$$x = 45 \sin\alpha \cos\alpha$$

if $\sin\alpha = \frac{2\sqrt{15}}{15}$, $\cos\alpha = \frac{\sqrt{165}}{15}$ using Pythagoras

$x = 45 \times \frac{2\sqrt{15}}{15} \times \frac{\sqrt{165}}{15}$

$x = 6\sqrt{11}\text{m}$ ✓ (show that question)

(c)

Prove true for $n = 0$

$$\begin{aligned} LHS &= \frac{1}{2^0}\tan\left(\frac{x}{2^0}\right) \\ &= \tan x \end{aligned}$$

$$\begin{aligned} RHS &= \frac{1}{2^0}\cot\left(\frac{x}{2^0}\right) - 2\cot(2x) \\ &= \cot x - 2\cot 2x \\ &= \frac{1}{\tan x} - \frac{2}{\tan 2x} \\ &= \frac{1}{\tan x} - \frac{2 - 2\tan^2 x}{\tan 2x} \\ &= \frac{2 - 2 + 2\tan^2 x}{2\tan x} \\ &= \tan x \\ &= LHS \quad ✓ \end{aligned}$$

Assume true for $n = k$

$$\sum_{r=0}^{k} \frac{1}{2^r}\tan\left(\frac{x}{2^r}\right) = \frac{1}{2^k}\cot\left(\frac{x}{2^k}\right) - 2\cot(2x)$$Prove true for $n = k + 1$

Required to Prove:

$$\sum_{r=0}^{k+1} \frac{1}{2^r}\tan\left(\frac{x}{2^r}\right) = \frac{1}{2^{k+1}}\cot\left(\frac{x}{2^{k+1}}\right) - 2\cot(2x)$$

Proof:

$$LHS = \sum_{r=0}^{k+1} \frac{1}{2^r}\tan\left(\frac{x}{2^r}\right)$$

$$= \sum_{r=0}^{k} \frac{1}{2^r}\tan\left(\frac{x}{2^r}\right) + \frac{1}{2^{k+1}}\tan\left(\frac{x}{2^{k+1}}\right)$$

(using assumption) $$= \frac{1}{2^k}\cot\left(\frac{x}{2^k}\right) - 2\cot 2x + \frac{1}{2^{k+1}}\tan\left(\frac{x}{2^{k+1}}\right) \quad \checkmark$$

$$= \frac{1}{2^{k+1}}\left(2\cot\left(\frac{x}{2^k}\right) + \tan\left(\frac{x}{2^{k+1}}\right)\right) - 2\cot 2x$$

Let $y = \dfrac{x}{2^k}$

$$= \frac{1}{2^{k+1}}\left(2\cot(2y) + \tan y\right) - 2\cot 2x$$

$$= \frac{1}{2^{k+1}}\left(\frac{2\cos(2y)}{\sin(2y)} + \tan y\right) - 2\cot 2x$$

$$= \frac{1}{2^{k+1}}\left(\frac{2\cos^2 y - 2\sin^2 y}{2\sin y\cos y} + \tan y\right) - 2\cot 2x$$

$$= \frac{1}{2^{k+1}}\left(\cot y - \tan y + \tan y\right) - 2\cot 2x$$

$$= \frac{1}{2^{k+1}}\cot\left(\frac{x}{2^{k+1}}\right) - 2\cot 2x$$

$= RHS$ Hence true for all $n \geq 0$ by induction ✓

(d) Roots of $8x^2 - 5x + a = 0$ are $\sin\theta$ and $\cos 2\theta$

$$\sin\theta + \cos 2\theta = \frac{5}{8} \qquad (1) \qquad \text{(from sum of roots)}$$

$$\sin\theta\cos 2\theta = \frac{a}{8} \qquad (2) \qquad \text{(from product of roots)} \quad \checkmark \text{ (both)}$$

(1) $$8\sin\theta + 8\cos 2\theta - 5 = 0$$

$$8\sin\theta + 8\left(1 - 2\sin^2\theta\right) - 5 = 0$$

$$-16\sin^2\theta + 8\sin\theta + 3 = 0$$

$$16\sin^2\theta - 8\sin\theta - 3 = 0$$

$$(4\sin\theta - 3)(4\sin\theta + 1) = 0$$

$$\sin\theta = \tfrac{3}{4} \qquad \text{or } \sin\theta = -\tfrac{1}{4} \quad \checkmark$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$= -\tfrac{1}{8} \qquad \text{or } = \tfrac{7}{8}$$

(2) $$\sin\theta\cos 2\theta = \tfrac{a}{8}$$

$$\tfrac{3}{4} \times -\tfrac{1}{8} = \frac{a}{8} \qquad \text{or } -\tfrac{1}{4} \times 78 = \tfrac{a}{8}$$

$$a = -\tfrac{3}{4} \qquad \text{or } a = -\tfrac{7}{4} \quad \checkmark$$

End of Solutions